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March 25, 1974

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Abstract

In this paper, we develop a model for a monopolist in a dynamic market who tries to maximize his overall profits over a certain planning horizon. Conditions for his optimal supply curve for both finite and infinite horizon problems are obtained and some interesting properties of the optimal paths are explored. A special (linear-quadratic) case is then treated and explicit characterizations of the optimal supply curve and the market price-supply behaviors are obtained.

[†]This work was supported in part by the Joint Services Electronics Program under Contract No. DAAB-07-72-C-0259 and in part by NSF under Grant No. GK 36276 with the Coordinated Science Laboratory, and in part by projects No. 05-382 and 389 with the USDA and the Agricultural Experiment Station at the University of Illinois.

Optimum Supply Control of a Monopolist in a Dynamic Market

M. Simaan and T. Takayama*

Introduction

Since the time Cournot investigated the economic rationale of firms in various markets, many economists have touched on the subject of "monopoly" to varying extents and depths. Analytical framework of the theory of monopoly has remained mainly static; Marshall (1920) in his Principles of Economics, Chapter XIV; Zeuthen (1955) in Part Four of his Economic Theory and Method, and Malinvaud (1971) on pages 70-75 in his Lectures on Microeconomic Theory, to mention only a few, all developed their theory in a static framework. Obviously, "monopoly" theory has its historical counterparts in real economic life. The most interesting examples are the Aluminum Company of America from 1888 until World War II (see Cohen and Cyert (1965), pp. 200-203), IBM in recent history, other big business firms mentioned elsewhere, and other government monopolies (for instance, the Japan Tobacco Monopoly Corporation). Except for those government monopoly cases, the firms referred to above seem to be under a continuous threat from potential newcomers in both domestic and international markets.

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Thus, for these private industries, any monopoly theory, whether static or dynamic in isolation of competitive firms may prove ineffective in either explaining the monopolist's behavior or providing quantitative information for the improvement of the monopolist's performance.

Irrespective of the above observation, in this paper we plan to develop a model of a monopolist who is completely free from the threat of his potential or imaginary competitors, and who tries to maximize his profit over his planning horizon $[0,T)$ where T may be either finite or infinite. It is interesting to note that as early as 1924, several references have been made in the mathematical economics literature to dynamic monopoly models such as the models developed and analyzed by Evans (1924), and Tintner (1937). However, due mainly to the lack of well developed dynamic optimization theory, most of these models were left without thorough analysis. In this paper, we plan to study both qualitative and quantitative aspects of our dynamic monopoly model by effectively exploiting techniques developed in optimal control theory. Another aspect that interests us is a similarity between our dynamic monopoly model and a dynamic economic planning model developed by Arrow (1968) in the same spirit as ours. Thus, a micro-economic model such as a monopoly model will find a way of dynamizing itself in the framework of optimal control theory.

In our model, we assume that the monopolist is placed in an environment where a dynamic demand function, instead of a static demand function, of the form:

$$\dot{p}(t) = f(p(t),x(t))$$

leads the firm to draw a rationale or optimal supply program. In the above differential equation,[1] $p(t)$ and $x(t)$ are the price and consumption quantities at time t respectively. In the first section of this paper, we develop this model and formulate the profit maximization problem that the monopolist faces in the market. In the second section, necessary conditions for the optimal supply quantities are obtained and some important properties of the finite and infinite horizon solutions are derived. In this development we follow closely Simaan and Takayama (1974). In the third section of this paper, we develop a special but easily tractable model in which the dynamic demand function is linear and the total production cost function is quadratic (Linear-Quadratic model) and show some of the interesting properties of the optimal trajectories in both finite and infinite horizons. Finally, in the fourth section we solve an example problem and in the last section we summarize the spectrum of results.

1. A General Dynamic Monopolist Model

In this paper, we consider a monopolist who at time t of a certain planning horizon $[0,T)$ produces a single commodity at the rate $x(t)$ and incurs a total production cost of

$$(1.1) \qquad TC = g(x(t))$$

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[1] For early development of different types of economic dynamics see Zeuthen (1955), Chapter 23, and Frisch referenced therein.

where $g(\cdot)$ is a convex function, at least twice differentiable and having a minimum at $x = 0$. The commodity is then sold at a price $p(t)$ which is determined dynamically in the market, to be defined later, and the monopolist secures the total revenues of

$$\text{(1.2)} \qquad TR = p(t)x(t)$$

and a total profit at time t of

$$\text{(1.3)} \qquad TP = p(t)x(t) - g(x(t)).$$

In contrast to the static monopolist model where the price $p(t)$ is instantaneously related to the production level $x(t)$ through a static demand function of the form $h(p(t)) - x(t) = 0$; we shall assume in this paper that the monopolist faces a dynamic market where the price at time t is determined through a dynamic demand function of the form

$$\text{(1.4)} \qquad \dot{p}(t) = \frac{dp(t)}{dt} = f(p(t),x(t)) \quad , \; p(0) = p_o$$

where p_o is the initial price of the commodity at time $t = 0$, the start of the planning horizon. Equation (1.4) essentially says that at each time t; the rate of change of the price $p(t)$ depends on the price level and consumption rate ($\equiv$ production rate, in our model as in Evans (1924)) at that particular time t. Stated in different terms (1.4) relates the price at time t, to the initial price p_o and to the entire history of consumption (supply or production) function $x(\tau)$ for τ in the interval of time $[0,t)$. Functionally this can be written as

(1.5) $$p(t) = \phi(p_o\ ;\ x(\tau),\ \tau \in [0,t))$$

where ϕ is the trajectory of the solution[2] of (1.4) for a given p_o and supply function $x(t)$. There are various assumptions that $f(p,x)$ must satisfy in order for (1.4) to make sense as a demand function. These are:

(i) $f(p,x)$ must satisfy the usual conditions for existence and uniqueness of solutions of differential equations, and furthermore we assume that for each function $x(t) \geq 0$ defined over a certain planning horizon $[0,T)$, where T may be infinite, (1.4) has a solution $p(t) \geq 0$ for all $t \in [0,T)$

(ii) $f(p,x)$ must be such that for each $x \geq 0$ there is a unique $p \geq 0$ such that $f(p,x) = 0$, and for each $p \geq 0$, $f(p,x)$ is concave and there is a unique $x \geq 0$ such that $f(p,x) = 0$

(iii) we also assume that $\frac{\partial f}{\partial p} \leq 0$, $\frac{\partial f}{\partial x} \leq 0$ for all $x \geq 0$ and $p \geq 0$, and that $f(p,x) = 0$ divides the positive p-x quadrant in two regions: (a) the upper region where $f(p,x) < 0$ and (b) the lower region where $f(p,x) > 0$.

The market "memory" in the price adjustment process (1.5) is essentially what makes the dynamic market different from the static one. While sudden changes in the supply will cause sudden changes in the price in a static market (see Fig. 1(a)); they will only cause gradual, slow (or delayed)

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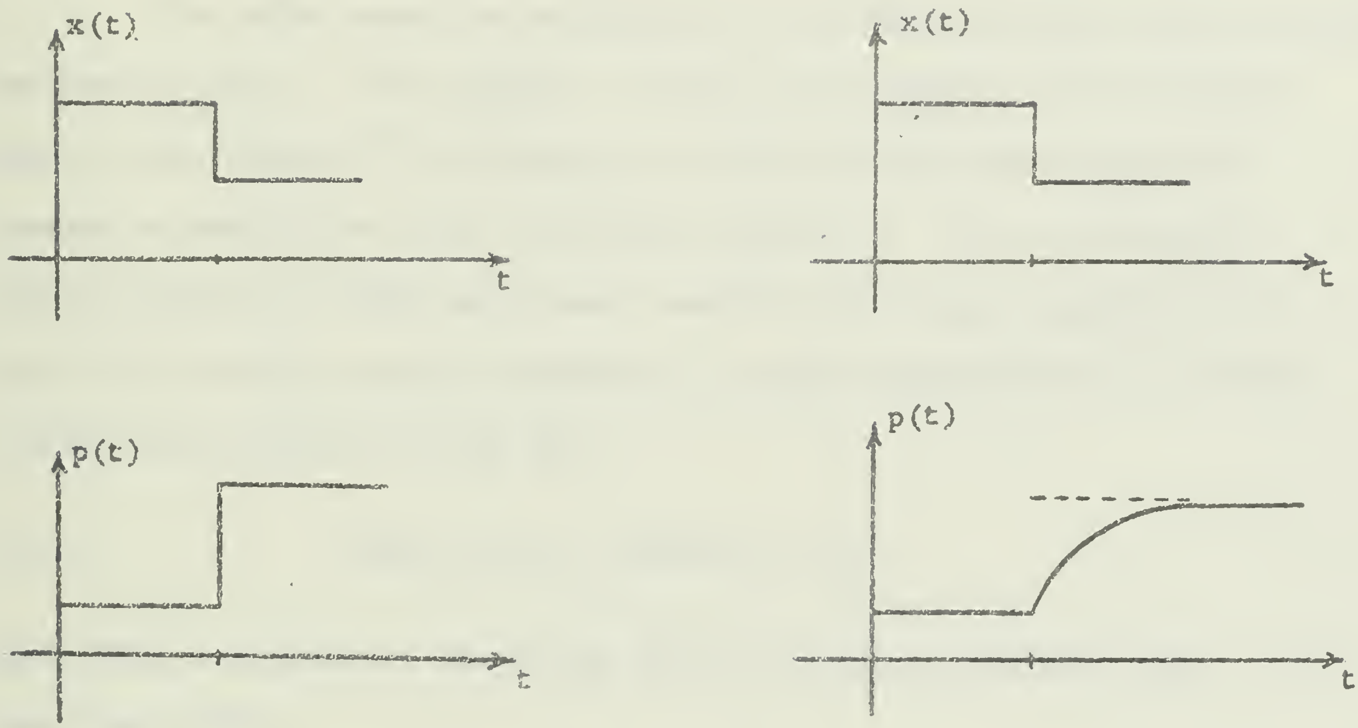
[2] Thus our actual dynamic demand function ϕ is a mapping from $R^+ \times C^+[0,t) \to R^+$, where $C^+[0,t)$ is the space of measurable nonnegative functions on $[0,t)$ and R^+ is the nonnegative part of the real line.

changes in the price in a dynamic market as illustrated[3] in Fig. 1(b) below:



(a) Static Market (b) Dynamic Market

Fig. 1. Market behavior under sudden changes in supply.

Thus in a dynamic market, it "takes time" for the price to adjust itself when there are time variations (not necessarily sudden as in Fig. 1) in the consumption or supply rate. Hence, the monopolist does not enjoy an instantaneously responsive market, and he is put in a situation where he

[3] The problem of identifying this dynamic demand function (i.e. $\dot{p} = f(p,x)$) from market data is an interesting problem by itself (for instance see Athans (1973) and Mendel (1973)) but will not be treated in this paper. In this paper we assume that the function $f(p,x)$ is known to the monopolist.

has to plan his supply, knowing that variations in it could cause variations in the price that will propagate over a certain period of time in the future.

The price-supply relationship of (1.4) can be studied in the p-x plane as shown in Fig. 2. For instance, if $x(t) = \bar{x}$ = constant, then the price moves in the direction[4] stipulated by the sign of $f(p,\bar{x})$ and eventually reaches an equilibrium value $\bar{p}$ such that $f(\bar{p},\bar{x}) = 0$. Thus, the dynamic demand function (or price adjustment function, the closest terminology we can find in static stability argument of a general equilibrium (see Nikaido (1970)) may be written in the form

(1.6) $$f(p(t),x(t)) = G(h(x(t)) - p(t))$$

where $G(u)$ is a monotone increasing function of u and satisfies (see Samuelson (1947))

(1.7) $$G(0) = 0 \quad \text{and} \quad \frac{dG(u)}{du} > 0 \quad \forall u \in R.$$

As mentioned earlier in this paper, the monopolist is assumed to be manipulating the supply function $x(t)$. We shall assume that his objective is to maximize his total profits over his planning horizon $[0,T)$:

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[4]Lindahl, in the framework of a discrete dynamics or "period analysis" called this type of process "disequilibrium method" in comparison with the Marshallian "equilibrium method." However, in our continuous dynamics, these two methods turn out to be identical if one carefully examines the content of the following development (see Baumol (1970), pp. 127-141).

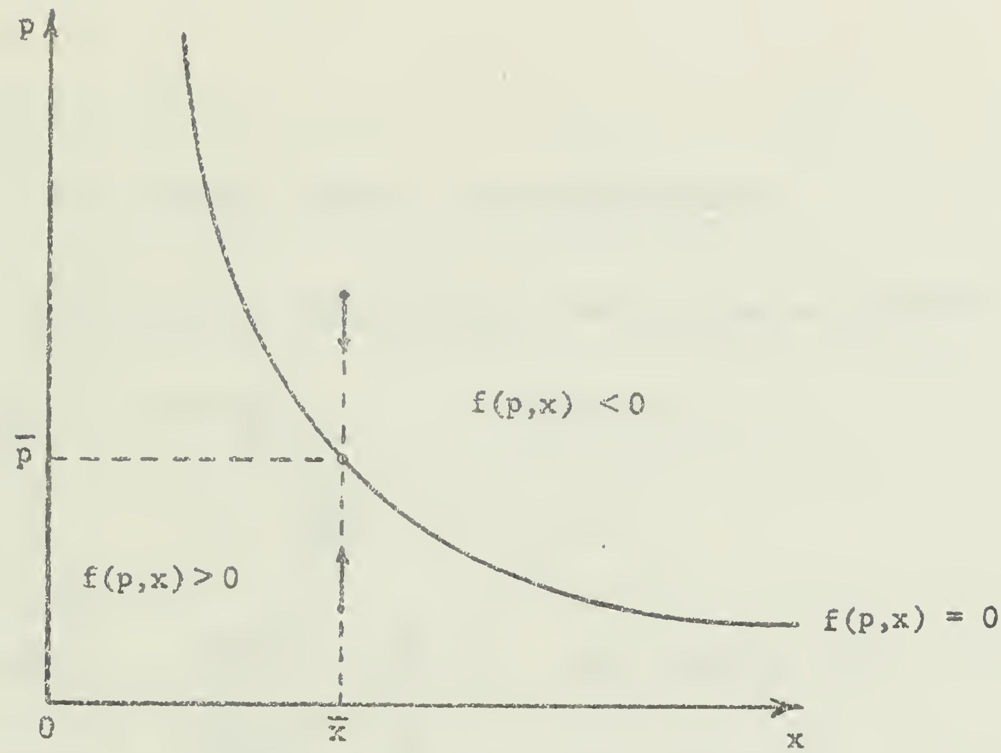

Fig. 2. Price variations for a fixed supply.

$$\pi(p_o, x(t)) = \int_0^T e^{-rt}(p(t)x(t) - g(x(t)))dt \tag{1.8}$$

where r is a suitable discount rate. Equation (1.8) explicitly defines the profits as a function of the initial price p_o and the supply $x(t)$ for all $t \in [0,T)$ which is obvious in view of the dynamics of the market (1.4).

2. The Optimal Supply Function

The problem of profit maximization (1.8) subject to the market dynamics (1.4) can be solved by applying well known results in optimal control theory (see Pontryagin et al. (1962)). First the Hamiltonian is

defined by

(2.1) $$H = p(t)x(t) - g(x(t,)) + \lambda(t)f(p(t),x(t)).$$

The necessary conditions for optimality are then obtained as follows:

$$(2.2)\quad \begin{cases} \text{(i)} & \dot{p} = f(p,x) \qquad , \; p(0) = p_0 \\ \text{(ii)} & \dot{\lambda} = (r - \frac{\partial f}{\partial p})\lambda - x \qquad , \; \lambda(T) = 0 \\ \text{and}^5 & \\ \text{(iii)} & p - \frac{dg(x)}{dx} + \lambda \frac{\partial f}{\partial x} = 0 \quad \text{for } x(t) \geq 0 \text{ , or} \\ \text{(iii)}' & p - \frac{dg(x)}{dx} + \lambda \frac{\partial f}{\partial x} < 0 \quad \text{for } x(t) = 0 \text{ .} \end{cases}$$

Thus for each p_0, the optimal supply function $x^*(t)$ can be obtained by solving (2.2). There are several properties that follow from these conditions, and these are summarized below.

Proposition 1: The costate variable $\lambda(t)$ satisfies $\lambda(t) \geq 0$ and it follows that:

(2.3) $$p(t) \geq \frac{dg(x(t))}{dx(t)} \qquad \forall t \in [0,T),$$

and equality holds at $t = T$.

Proof: The proof follows easily by contradiction. Suppose $\lambda(t) < 0$, then since $\lambda(T) = 0$ there must exist a $t_1 \in [t,T)$ such that $\lambda(t_1) < 0$ and

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[5] Note that the conditions on $f(p,x)$ and $g(x)$ insure that H is concave in x.

$\dot{\lambda}(t_1) > 0$. However, from (2.2 ii) we see that this is impossible. Thus $\lambda(t) \geq 0$, and (2.3) follows from (2.2 iii), and the fact that $\lambda(T) = 0$.

Irrespective of the differences between the static and dynamic models there is, as Tintner (1937) pointed out earlier, a remarkable resemblance between the static profit maximizing condition (marginal revenue = marginal cost) and its dynamic counterpart (2.2 iii):

$$\text{(2.4)} \qquad \text{marginal cost} = \frac{dg(x)}{dx} = p + \lambda \frac{\partial f}{\partial x} = \text{marginal revenue.}$$

The end of horizon condition $\lambda(T) = 0$ which leads to

$$\text{(2.5)} \qquad p(T) = \frac{dg(x(T))}{dx(T)}$$

protrays the benevolent monopolist profit maximization condition (Takayama and Judge (1971), pp. 225-230). This is due to our assumption in (1.8) that the terminal profit or salvage value is independent of $p(T)$; however if a terminal profit $F(p(T))$ is introduced in (1.8) then the boundary condition of (2.2 ii) will become

$$\text{(2.6)} \qquad \lambda(T) = \frac{dF(p(T))}{dp(T)} .$$

One of the interesting aspects of the above analysis is to examine the nature of the price-supply dynamics in the p-x plane. By simple differentiation and algebraic manipulations of (2.2 i, ii and iii) $\lambda(t)$ can be eliminated and the following system of differential equations can be easily obtained:

$$(2.7)\quad \begin{cases} \dot{p} = f(p,x) & p(0) = p_o \\ \dot{x} = h(p,x) & p(T) = \dfrac{dg(x(T))}{dx(T)} \end{cases}$$

where

$$h(p,x) = \frac{f\frac{\partial f}{\partial x} - x(\frac{\partial f}{\partial x})^2 - (p - \frac{dg}{dx})(f\frac{\partial^2 f}{\partial x \partial p} + (r - \frac{\partial f}{\partial p})\frac{\partial f}{\partial x})}{(p - \frac{dg}{dx})\frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x}\frac{d^2 g}{dx^2}}.$$

Let us assume[6] that the equation $h(p,x) = 0$ generates a single-valued function $p = H(x)$ such that at each pair $(\tilde{p},\tilde{x})$ in the (p,x) plane, $h(\tilde{p},\tilde{x})$ is positive if $\tilde{p} < H(\tilde{x})$ and $h(\tilde{p},\tilde{x})$ is negative if $\tilde{p} > H(\tilde{x})$. We can now draw a phase-plane diagram for (2.7) as shown in Fig. 3. For each initial price p_o, (2.7) can be solved and the corresponding optimal initial supply quantity $x(0) = x_o$ can be obtained. This, therefore, generates an optimal initial manifold from which all optimal paths are started.[7] On the other hand, as the terminal time T is reached, all optimal paths must terminate on the terminal manifold given by (2.5). Several such optimal paths are illustrated in Fig. 3.

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[6] The fact that these assumptions indeed hold for the general model (2.7) is very involved and lengthy to prove; however they will be shown to hold for the special case of linear demand and quadratic cost functions to be discussed in the following section.

[7] An interesting question from the monopolist point of view may be raised at this point: Is there an initial price p_o^* that gives the monopolist the maximum possible profit? The answer to this question is straightforward: select p_o^* such that $\lambda(0) = 0$ (which is obvious since $\lambda(0) = \frac{\partial\pi(p_o,x(t))}{\partial p_o}$). This condition implies (2.2 iii) that $p_o = \left.\frac{dg(x)}{dx}\right|_{t=0}$, and it essentially means that the optimal initial price, (if it exists) must be the intersection point of the initial and terminal manifolds (Fig. 3).

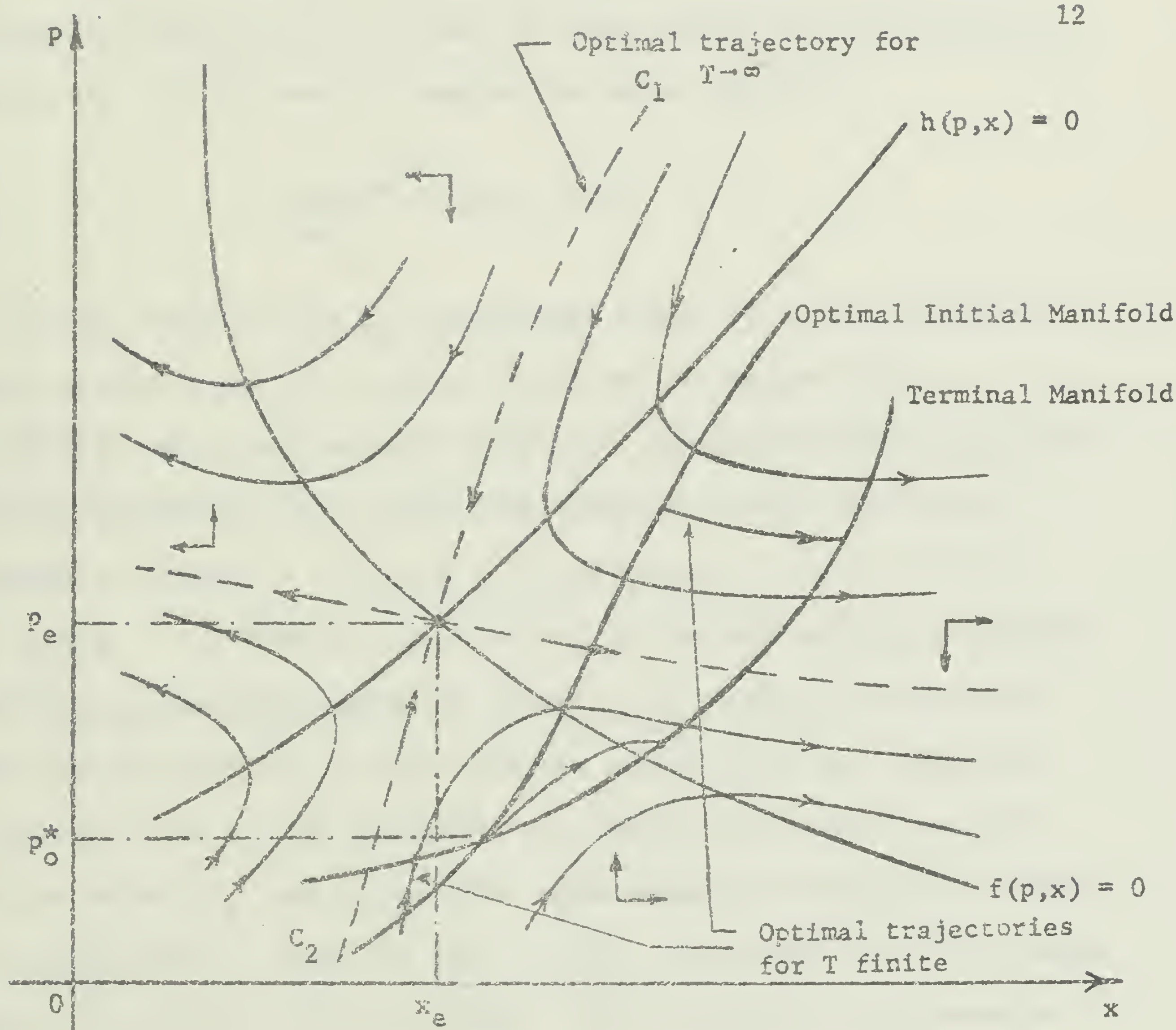

Fig. 3. Phase plane trajectories.

We now discuss the infinite horizon case where $T \to \infty$. The conditions of optimality are the same as (2.2) except that the boundary condition for (2.2 ii) must be changed (see Arrow (1968)) to

$$\lim_{t \to \infty} e^{-rt} \lambda(t) p(t) = 0. \tag{2.8}$$

Thus for each initial price p_o, the optimal system (2.2) with (2.8) as boundary condition for $\lambda(t)$, can be solved for the optimal path $\{p^*(t), x^*(t)\}$, which will satisfy (2.8) if an equilibrium point (p_e, x_e) is eventually reached as $t \to \infty$. Thus this equilibrium point must be the intersection of $f(p,x) = 0$ (i.e. $\dot{p} = 0$) and $h(p,x) = 0$ (i.e. $\dot{x} = 0$) as shown in Fig. 3. Obviously, there is only one optimal trajectory passing through $\{p_e, x_e\}$ and this trajectory (curve $C_1 C_2$ in Fig. 3) is also the optimal initial manifold for this infinite horizon problem. Hence for each initial price p_o, the monopolist must adjust his initial supply to be on the curve C_1C_2, and as the price moves upwards or downwards, he must keep on adjusting his supply so that at each t, the point $\{p(t), x(t)\}$ stays always on C_1C_2 until $\{p_e, x_e\}$ is reached. This trajectory will guarantee the monopolist maximum profit. Finally, it may be worthwhile to mention at this point, the resemblance between this dynamic monopolist profit maximizing behavior and the dynamic economic planning model discussed in Arrow (1968). We will sharpen these results in the next section and fully investigate various aspects of the initial and terminal manifolds and optimal trajectories of our linear-quadratic monopoly model .

3. The Linear Demand and Quadratic Cost Monopolist Model

An interesting special case of the previous analysis which leads to an analytically tractable solution is when the demand function (1.4) is linear and the production cost function (1.1) is quadratic. That is when[8]

$$\dot{p} = c - ap - bx \qquad , \; p(0) = p_o \tag{3.1}$$

and

$$g(x) = \frac{1}{2}\alpha x^2. \tag{3.2}$$

The parameters c, a, b and α are assumed to be known and positive. The profits over the time horizon $[0,T)$ are then:

$$\pi(p_o, x(t)) = \int_0^T e^{-rt}[p(t)x(t) - \frac{1}{2}\alpha x^2(t)]dt \tag{3.3}$$

and the necessary conditions[8] (2.2) can be easily written as:

$$\begin{cases} \text{(i)} & \dot{p} = c - ap - bx & p(0) = p_o \\ \text{(ii)} & \dot{\lambda} = (r+a)\lambda - x & \lambda(T) = 0 \\ \text{(iii)} & p - \alpha x - b\lambda = 0 & \text{for} \quad 0 \le x \le \frac{c}{b} \\ \text{(iii)'} & p - \alpha x - b\lambda < 0 & \text{for} \quad x = 0 \\ \text{(iii)''} & p - \alpha x - b\lambda > 0 & \text{for} \quad x = \frac{c}{b} \end{cases} \tag{3.4}$$

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[8]Note that the function $f(p,x) = c - ap - bx$ satisfied the conditions in Section 1 only in a compact rectangle in the p-x plane defined by $0 \le p \le \frac{c}{a}$ and $0 \le x \le \frac{c}{b}$. In the necessary conditions of optimality (2.2) account must then be taken of the constraint $x(t) \le \frac{c}{b}$ in addition to $x(t) \ge 0$. The constraint that $0 \le p(t) \le \frac{c}{a}$ will then be automatically satisfied in view of the solution of (3.1):

$$p(t) = \frac{c}{a} - (\frac{c}{a} - p_o)e^{-at} - be^{-at}\int_0^t e^{a\tau}x(\tau)d\tau,$$

and for $0 \le x(t) \le \frac{c}{b}$, and need not be accounted for in the necessary conditions.

By exploiting the results obtained for our general dynamic monopoly model, let us study the dynamic properties of the phase-plane diagram based on (2.7), which now can be written as:

$$(3.5)\quad \begin{cases} \dot{p} = f(p,x) = c - ap - bx & p(0) = p_0 \\ \dot{x} = h(p,x) = \dfrac{c}{\alpha} - \dfrac{r+2a}{\alpha}\, p + (r+a)x & , \; x(T) = \dfrac{1}{\alpha}\, p(T) \end{cases}$$

by eliminating $\lambda(t)$ from (3.4)(i), (ii) and (iii). The phase plane is now clearly divided into four regions by the two lines $f(p,x) = 0$ and $h(p,x) = 0$ and the general discussions made on Fig. 3 apply exactly to this case. The phase-plane and trajectories are shown in Fig. 4. Let us first discuss the finite horizon problem.

3.a. The Finite Horizon Case

The terminal manifold in this case is given by the line $x(T) = \frac{1}{\alpha}\, p(T)$. In order to determine which trajectory is optimal for a given initial p_0, it is necessary to solve the two point boundary value system given by (3.5) (or (3.4)). However since our system is of the linear quadratic type, a transformation of the Riccati type, well known in optimal control theory (see Athans and Falb (1966)), can reduce it to a single point boundary value problem. We shall express this transformation as follows

$$(3.6)\qquad x(t) = K(t)p(t) + E(t)$$

where $K(t)$ and $E(t)$ are functions of time to be computed. This transformation

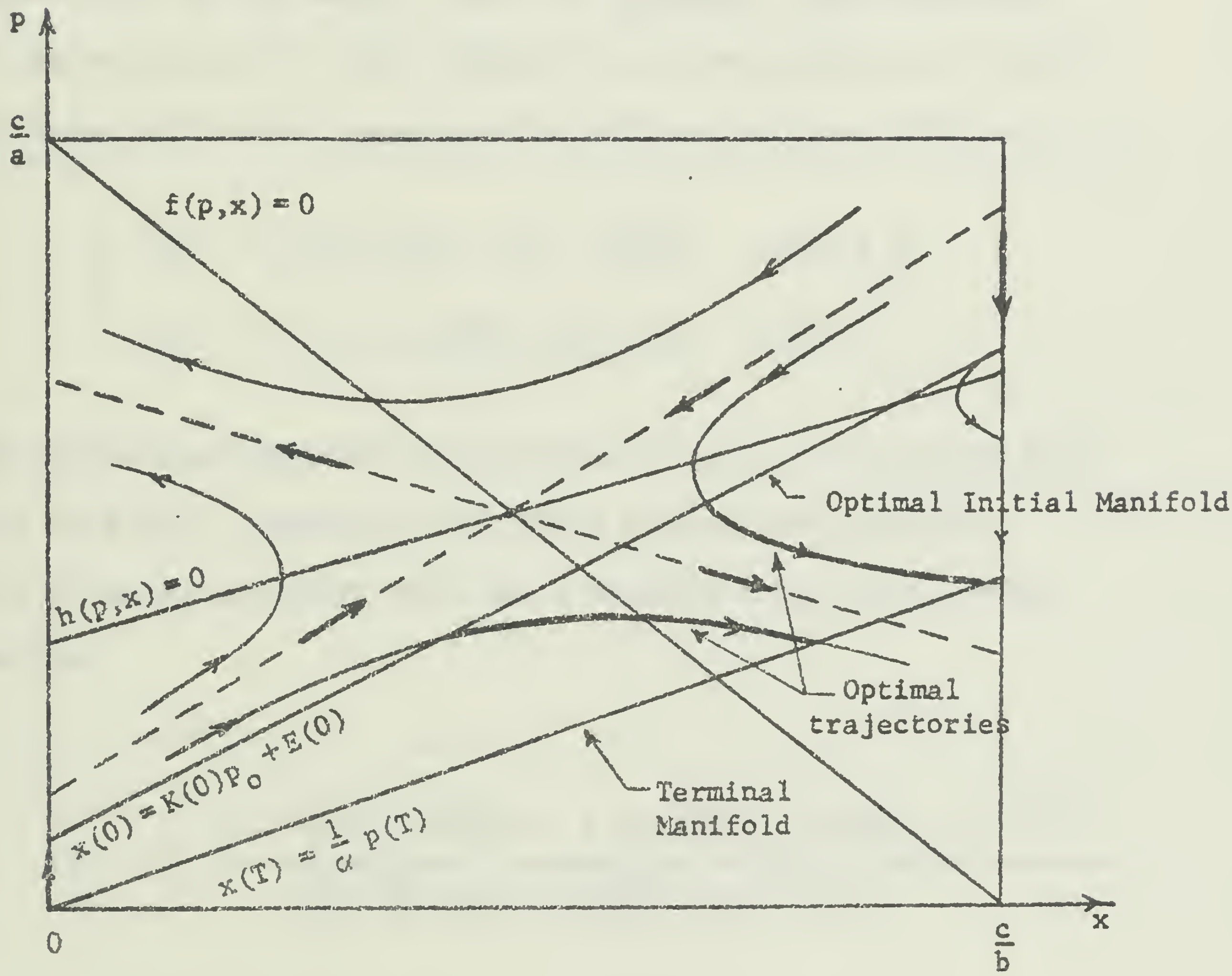

Fig. 4. Phase plane for the Linear Quadratic Problem.

has a physical significance; it expresses the supply at time t as a function of the price of time t. This will then be the optimal supply curve that will determine the monopolist's output based on the current price prevailing in the market, that is a "feedback" supply function. Upon differentiating (3.6) with respect to time and making use of (3.5), the following differential equations for K(t) and E(t) are obtained:

$$(3.7)\qquad \begin{cases} \text{(i)} & \dot{K} = K(r+2a) + bK^2 - \dfrac{r+2a}{\alpha}, \quad K(T) = \dfrac{1}{\alpha} \\ \text{(ii)} & \dot{E} = (r+a+bK)E + c\left(\dfrac{1}{\alpha} - K\right), \quad E(T) = 0. \end{cases}$$

Thus by solving (3.7) backward in time from $t = T$ to $t = 0$, we get K(t) and E(t) $\forall t \in [0,T)$. Equation (3.7)(i) is a first order quadratic equation of the Riccati-type which has a solution (obtained by simple integration):

$$(3.8)\qquad K(t) = \frac{1}{b}\;\frac{\left(\beta - \frac{r+2a}{2}\right)\left(\frac{b}{\alpha} + \frac{r+2a}{2} + \beta\right) + \left(\beta + \frac{r+2a}{2}\right)\left(\frac{b}{\alpha} + \frac{r+2a}{2} - \beta\right)e^{-2\beta(T-t)}}{\left(\frac{b}{\alpha} + \frac{r+2a}{2} + \beta\right) - \left(\frac{b}{\alpha} + \frac{r+2a}{2} - \beta\right)e^{-2\beta(T-t)}}$$

where

$$\beta = \sqrt{\frac{(r+2a)^2}{4} + \frac{(r+2a)b}{\alpha}}\,.$$

Equation (3.7)(ii) is linear time-varying, and has a solution of the form:

$$(3.9)\qquad E(t) = -\int_t^T c\left(\frac{1}{\alpha} - K(\tau)\right)e^{\int_T^t (r+a+bK(\mu))d\mu}\,d\tau .$$

By simple algebraic manipulations, it is easy to obtain the following bounds on $K(t)$ and $E(t)$

$$(3.10)\qquad \begin{cases} \text{(i)} \quad 0 < K(t) \leq \frac{1}{\alpha} \\ \text{(ii)} \quad E(t) \leq 0 \, . \end{cases}$$

Equations (3.10)(i) and (3.6) reveal an interesting property of the optimal marginal supply function; namely:

$$(3.11)\qquad 0 < \frac{dx(t)}{dp(t)} = K(t) \leq \frac{1}{\alpha}$$

whenever (3.4)(iii) holds. From the above analysis, the optimal initial manifold is therefore a line of the form

$$(3.12)\qquad x(0) = K(0)p_o + E(0),$$

and every optimal trajectory must start on this manifold, satisfy (3.6) at each t and move along the solution of:

$$(3.13)\qquad \dot{p}(t) = (c - bE(t)) - (a + bK(t))p(t) \qquad , p(0) = p_o$$

and finally terminate on the line

$$(3.14)\qquad x(T) = \frac{1}{\alpha}\, p(T).$$

It is important at this stage to mention the following three interesting observations:

(a) $K(0)$ and $E(0)$ in (3.12) are both functions of T; and a simple analysis of (3.8) reveals that the larger is T, the smaller is its corresponding $K(0)$; i.e. if $T_1 > T_2$ then $K_1(0) < K_2(0)$.

(b) As indicated in (3.4)(iii)' and (iii)", the initial manifold (3.12) holds for all initial prices p_o such that the condition $0 \leq x(0) \leq \frac{c}{b}$ is satisfied. In other words such that:

$$- \frac{E(0)}{K(0)} \leq p_o \leq \frac{c/b - E(0)}{K(0)} \; . \tag{3.15}$$

If however (3.15) is not satisfied for a certain p_o, then according to (3.4)(iii)' and (iii)" the supply function must be kept at either $x(t) = 0$ or $x(t) = \frac{c}{b}$ according to the following rule:

$$\begin{cases} \text{(i)} \quad x(t) = 0 & \text{if} \quad p(t) < - \dfrac{E(t)}{K(t)} \\ \text{(ii)} \quad x(t) = \dfrac{c}{b} & \text{if} \quad p(t) > \dfrac{c/b - E(t)}{K(t)} \end{cases} \tag{3.16}$$

which will cause the price to rise for case (i) and to fall for case (ii) until a time t_1 is reached where the price $p(t_1)$ is on the manifold (3.12), and then the optimal trajectory will continue according to (3.12) until the terminal manifold is reached. This situation is clearly illustrated in Fig. 4.

(c) It is interesting to point out that a fixed end-point version of this finite horizon linear quadratic problem was treated by Evans as early as 1924 using calculus of variations techniques. Even though his treatment focused mainly on obtaining the optimal price trajectory by solving a second order differential equation, it may be worthwhile

to point out how the present treatment can be adapted to generate Evans' optimal trajectory. If we assume that the price at time T is fixed at $p(T) = p_1$ and if $r = 0$ (as in Evans (1924)) then an optimal trajectory inside $0 \leq x \leq \frac{c}{b}$ and $0 \leq p \leq \frac{c}{a}$ must satisfy the necessary conditions (3.4):

$$(3.17) \quad \begin{cases} \text{(i)} & \dot{p} = c - ap - bx & p(0) = p_o \quad , \; p(T) = p_1 \\ \text{(ii)} & \dot{\lambda} = a\lambda - x & \lambda(T) \text{ is free} \\ \text{(iii)} & p - \alpha x - b\lambda = 0 \quad . & \end{cases}$$

By differentiating (3.17)(i) and eliminating $\lambda(t)$ and $x(t)$, we can easily obtain Evans' equation for the optimal price trajectory:

$$(3.18) \qquad \ddot{p}(t) - a(a + \frac{2b}{\alpha})p + (a + \frac{b}{\alpha})c = 0.$$

The solution of this equation exhibits two exponential modes and is of the form

$$(3.19) \qquad p(t) = \bar{p} + C_1{}^{mt} + C_2\, e^{-mt}$$

where
$$m = \sqrt{a(a + \frac{2b}{\alpha})} \quad \text{and} \quad \bar{p} = \frac{c}{m^2}\,(a + \frac{b}{\alpha});$$

and C_1 and C_2 are constants that can be obtained from the conditions $p(0) = p_o$ and $p(T) = p_1$.

These results are nicely reflected on the phase plane diagram of Fig. 4, where an optimal trajectory is now connecting two horizontal lines at p_o and p_1. The shape of this optimal trajectory is easily revealed from

the location of the lines p_o and p_1 on the diagram. We will now discuss the infinite horizon problem.

3.b. The Infinite Horizon Case

As $T \to \infty$, it can be easily shown that the functions K(t) and E(t) in (3.6) will converge to some constant values ($\dot{K}$ and $\dot{E}$ in (3.7) will tend to zero) say $\bar{K}$ and $\bar{E}$. Following our discussion in Section 2, the corresponding trajectory will be optimal if an equilibrium level (p_e, x_e) is eventually reached. Thus upon solving the algebraic equations (3.7) after setting $\dot{K} = \dot{E} = 0$, the following solutions are obtained:

$$(3.20)\quad \begin{cases} \text{(i)} \begin{cases} \bar{K}_1 = -\dfrac{r+2a}{2b} + \dfrac{1}{2b}\sqrt{(r+2a)^2 + \dfrac{4b}{\alpha}(r+2a)} \\ \bar{K}_2 = -\dfrac{r+2a}{2b} - \dfrac{1}{2b}\sqrt{(r+2a)^2 + \dfrac{4b}{\alpha}(r+2a)} \end{cases} \\ \text{(ii)} \begin{cases} \bar{E}_1 = \dfrac{c(\bar{K}_1 - \frac{1}{\alpha})}{r+a+b\bar{K}_1} \\ \bar{E}_2 = \dfrac{c(\bar{K}_2 - \frac{1}{\alpha})}{r+a+b\bar{K}_2} \end{cases} \end{cases}$$

We now identify the pair $(\bar{K}_1, \bar{E}_1)$ as corresponding to the optimal path since the corresponding "feedback" supply law:

$$(3.21)\qquad x^*(t) = \bar{K}_1 p(t) + \bar{E}_1$$

leads to a price trajectory satisfying

$$(3.22)\qquad \dot{p} = (c - b\bar{E}_1) - (a + b\bar{K}_1)p \quad , \quad p(0) = p_o$$

and which (since $a + b\bar{K}_1 > 0$) as $t \to \infty$ reaches asymptotically the equilibrium price level

$$p_e = \frac{c - b\bar{E}_1}{a + b\bar{K}_1}$$

or

$$p_e = \frac{c}{a} \frac{r + a + \frac{b}{\alpha}}{r + a + \frac{b}{\alpha}\left(2 + \frac{r}{a}\right)} \tag{3.23}$$

The corresponding equilibrium supply level, easily computed from (3.21) is

$$x_e = \frac{c}{b} \frac{\frac{b}{\alpha}\left(1 + \frac{r}{a}\right)}{r + a + \frac{b}{\alpha}\left(2 + \frac{r}{a}\right)} \tag{3.24}$$

At this stage, we note that this equilibrium point (p_e, x_e) is also the intersection point of the two linear functions $f(p,x) = 0$ and $h(p,x) = 0$ given in (3.5).

On the other hand, the pair $(\bar{K}_2, \bar{E}_2)$ corresponds to the supply function

$$x(t) = \bar{K}_2 p_2(t) + \bar{E}_2 \tag{3.25}$$

which (since $a + b\bar{K}_2 < 0$) leads to an unstable price trajectory satisfying:

$$\dot{p} = (c - b\bar{E}_2) - (a + b\bar{K}_2)p_2 \qquad , \quad p(0) = p_o. \tag{3.26}$$

The phase-plane representation corresponding to this infinite horizon problem is shown in Fig. 5. The optimal supply curve is clearly composed of three parts:

[illegible]

$$[illegible]$$

[illegible]

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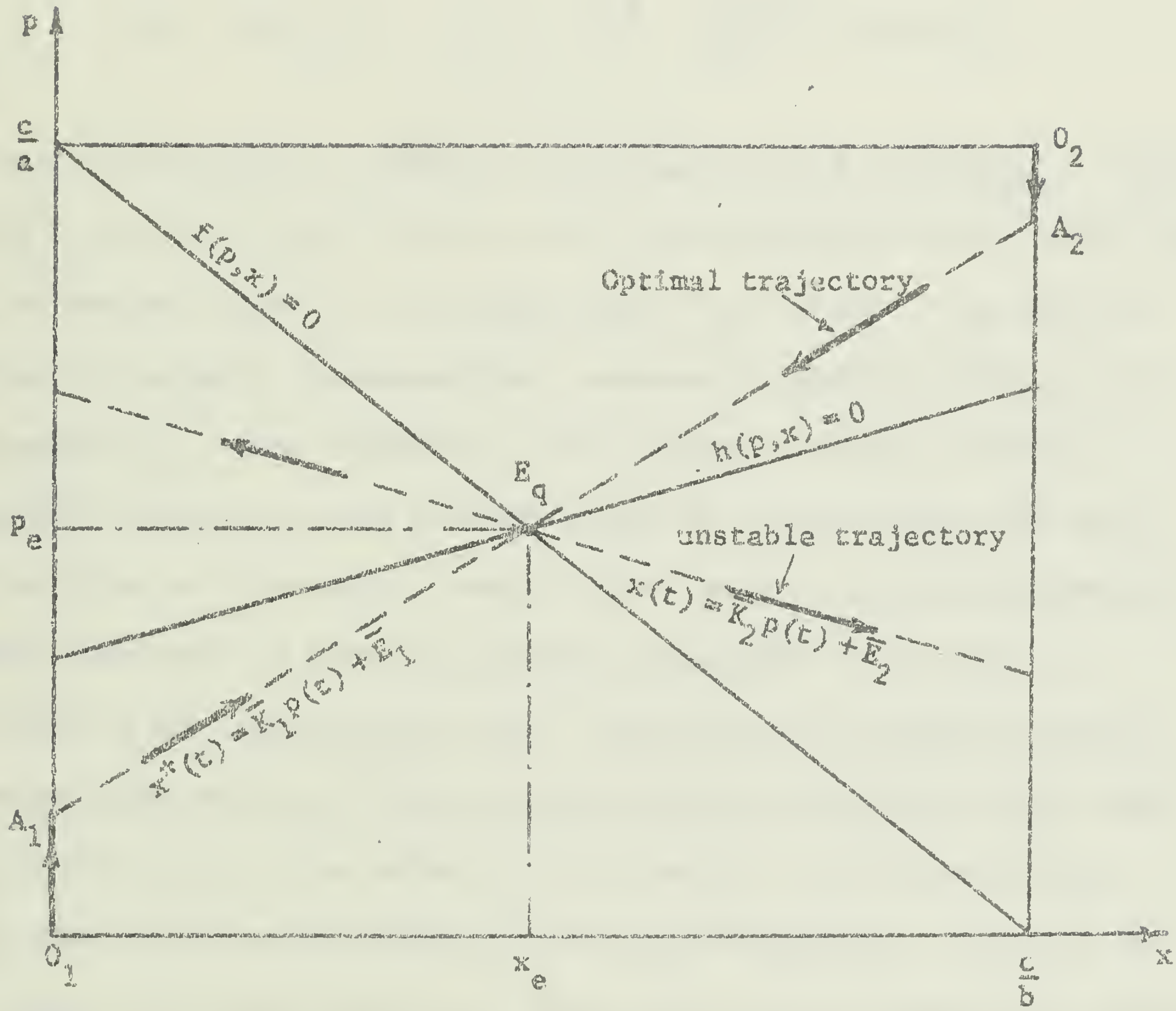

Fig. 5. Optimal supply curve for infinite horizon linear quadratic case.

$$(3.27)\begin{cases} \text{(i)} \quad x(t) = 0 & \text{if} \quad 0 \leq p(t) < -\dfrac{\bar{E}_1}{\bar{K}_1} \\ \text{(ii)} \quad x(t) = \bar{K}_1 p(t) + \bar{E}_1 & \text{if} \quad -\dfrac{\bar{E}_1}{\bar{K}_1} \leq p(t) \leq \dfrac{c/b - \bar{E}_1}{\bar{K}_1} \\ \text{(iii)} \quad x(t) = \dfrac{c}{b} & \text{if} \quad \dfrac{c/b - \bar{E}_1}{\bar{K}_1} < p(t) \leq \dfrac{c}{a} . \end{cases}$$

The optimal trajectory is shown by the segments $O_1A_1E_q$ and $O_2A_2E_q$. Points A_1 and A_2 are the "entry" points at which the monopolist should switch from zero or maximum supply to his linear supply rule (3.21). They have an interesting economic interpretation: Suppose the initial price p_o is on the segment O_1A_1, then according to the previous analysis, a profit maximizing monopolist would keep his supply at the zero level and wait for the price to rise until a certain level (point A_1) is reached when it becomes beneficial to place his output in the market and control it according to his linear supply curve. Similarly if the initial price p_o is larger than point A_2, then a profit maximizing monopolist would inject his maximum supply in the market and will switch to his linear supply curve when the price has fallen to the level of point A_2. As $t \to \infty$, in both cases, the optimal price and supply will reach the equilibrium level (p_e,x_e) and remain there as long as there are no external disturbances to the market. It is interesting to observe that the point (p_e,x_e) is a function only of the parameters a, b, c, r and α of the model; and that

$$(3.28) \qquad \frac{\partial p_e}{\partial r} = -\frac{c}{a} \; \frac{\dfrac{b^2}{\alpha^2 a}}{\left[r + a + \frac{b}{\alpha}(2 + \frac{r}{a})\right]^2} < 0$$

and

$$(3.29) \qquad \frac{\partial p_e}{\partial \alpha} = \frac{c}{a} \; \frac{\dfrac{b}{\alpha^2 a}(r+a)^2}{\left[r + a + \frac{b}{\alpha}(2 + \frac{r}{a})\right]^2} > 0.$$

The above expressions essentially confirm reasonable comparative dynamics conclusions that:

i) The equilibrium price is lower for a higher discount rate, and

ii) The equilibrium price will be higher if the cost of production is higher.

4. A Numerical Example

In this section, we illustrate the linear-quadratic infinite horizon solution by the following simple numerical example. Let

$$\dot{p} = 4 - 2p - x \qquad , \quad p(0) = p_0$$

and

$$\pi(p_0, x(t)) = \int_0^\infty e^{-0.1t}[p(t)x(t) - \frac{1}{8}x^2(t)]dt.$$

The parameter values for this problem are: $a = 2$, $b = 1$, $c = 4$, $\alpha = \frac{1}{4}$ and $r = 0.1$. From (3.20) we have $\bar{K}_1 = 2.50$ and $\bar{E}_1 = -0.13$; and the optimal supply function (3.27) is

$$\begin{cases} \text{(i)}\ \ x(t) = 0 & \text{if}\quad 0 \leq p(t) < 0.05 \\ \text{(ii)}\ \ x(t) = 2.5\, p(t) - 0.13 & \text{if}\quad 0.05 \leq p(t) \leq 1.65 \\ \text{(iii)}\ \ x(t) = 4 & \text{if}\quad 1.65 < p(t) \leq 2\ . \end{cases}$$

Finally the equilibrium level is given by:

$$p_e = 1.78$$

$$x_e = 1.63\ .$$

Conclusions

In this paper, an attempt has been made to formulate the profit maximizing monopoly problem within the framework of optimal control theory. It was assumed that the price-consumption (= supply) relationship in the market is governed by a dynamic demand function and that the objective of the monopolist is to maximize his total discounted profit over a certain time horizon that may be either finite or infinite. This model, in contrast with the static model, accounts for the dependence of the commodity price at future times on the current supply rate. Several interesting results that are not apparent from the static model have been obtained from this dynamic formulation. First, for the finite horizon problem, the existence of two manifolds in the p-x plane has been established: the initial optimal manifold, which is the starting point of every optimal trajectory and the terminal manifold on which every optimal trajectory must terminate. Furthermore, it was shown that the point of intersection of these two manifolds (if it exists) gives the best initial

starting price p_0^* for the monopolist in order to obtain his highest maximum profit. Second, for the infinite horizon problem, a behavior similar to Arrow's economic planning model (1968) has been pointed out and it was shown that all optimal paths must follow one optimal stable trajectory, in the p-x plane, which eventually converges to an equilibrium point (p_e, x_e).

The special, but highly operational case of linear demand and quadratic cost functions has then been treated in detail and the monopolist's optimal supply curve was shown explicitly to be composed of three sections depending on the current market price of the commodity. These sections are described as follows: (a) if the price is below a certain level the monopolist should cut his supply to zero; (b) if the price is above a certain level, then the monopolist should place his maximum supply capacity in the market and (c) if the price is between these two levels, he should use a linear supply curve. The optimal price-supply trajectories have been illustrated on the p-x plane for both the finite horizon case where a linear terminal manifold is reached and the infinite horizon case where an equilibrium point is eventually attained. In addition, some conclusions have been obtained with regards to the effect of the discount rate and the coefficient of the quadratic cost function on the equilibrium price level: a higher discount rate will lead to a lower equilibrium price level and a higher production cost will lead to a higher equilibrium price level.

We feel that in this paper we have explored various features of dynamic monopoly markets in the light of modern optimal control theory.

Developments in this direction naturally lead us towards dynamic duopoly, oligopoly, and perfect competition markets. A dynamic duopoly game has been formulated and solved for its dynamic Cournot solutions (see Simaan and Takayama (1974)) in the framework of differential game theory; however, due to their complex structures dynamic oligopoly and perfect competition models remain as challenging problems for future research in this area.

References

Arrow, K., "Application of Control Theory to Economic Growth," in Lectures in Applied Mathematics, Stanford University Press, 1968, pp. 85-119.

Athans, M. and P. L. Falb, Optimal Control; and Introduction to Theory and Its Applications, McGraw-Hill, 1966.

Athans, M., "The Importance of Kalman Filtering Methods for Economic Systems," paper presented at the Second Workshop on Stochastic Control and Economic Systems, Univ. of Chicago, June 7-9, 1973.

Baumol, W., Economic Dynamics, McMillan, 1970.

Cohen, K. J. and R. M. Cyert, Theory of the Firm: Resource Allocation in a Market Economy, Prentice-Hall, 1965.

Evans, G. C., "The Dynamics of Monopoly," American Mathematical Monthly, Vol. 31, No. 2, 1924, pp. 77-83.

Malinvaud, E., Lectures on Microeconomic Theory, North Holland, 1971.

Marshall, A., Principle of Economics, McMillan, eighth ed., 1920.

Mendel, J. M., Discrete Techniques of Parameter Estimation, Marcel Dekker, 1973.

Nikaido, H., Introduction to Sets and Mappings in Modern Economics, North Holland, 1970.

Pontryagin, L. S., V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mischenko, The Mathematical Theory of Optimal Processes, Interscience Publishers, 1962.

Samuelson, P. A., Foundation of Economic Analysis, Cambridge, 1947.

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Simaan, M. and T. Takayama, "Dynamic Duopoly Game: Differential Game Theoretic Approach," Faculty working paper #155, College of Commerce and Business Administration, Univ. of Illinois, 1974.

Takayama, T. and G. Judge, Spatial and Temporal Price and Allocation Models, North Holland, 1971.

Tintner, G., "Monopoly Over Time," Econometrica, Vol. 5, 1937, pp. 160-170.

Zeuthen, F., Economic Theory and Methods, Harvard, 1955.